MATH 147: GUIDELINES AND PRACTICE PROBLEMS FOR EXAM 2

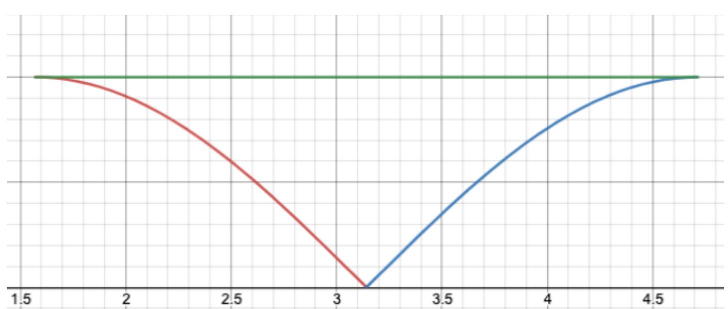
Topics covered on Exam 2.

(i) Double integrals via iterated integrals and Fubini's Theorem. Interchanging the order of integration.
(ii) Double integrals via polar coordinates.
(iii) Improper double integrals.
(iv) Various transformations of $\mathbb{R}^2$, their Jacobians and inverses, especially linear transformations, the one-to-one property.
(vi) Double integrals using the change of variables formula.
(vii) Triple integrals, Fubini's theorem, and changing the order of integration.
(viii) Various transformations of $\mathbb{R}^3$, their Jacobians, including spherical and cylindrical transformations.
(ix) Solving triple integrals with a change of variables formula, including spherical and cylindrical coordinates.
(x) Students should be able to state various definitions and answer true-false questions about topics covered since the first exam.

Practice problems.

1. OS Chapter 5: # 105: Find the volume under the graph of $z = x^3$ above the region D in the plane bounded by $x = \sin(y), x = -\sin(y), x = 1$, with $\frac{\pi}{2} \leq y \leq \frac{3\pi}{2}$.

Solution. Without loss of generality, we interchange the roles of x and y, so that we want $\int\int_D y^3\ dA$, with D pictured below.



,

where the brown line is that portion of $y = \sin(x)$ with $\frac{\pi}{2} \leq x \leq \pi$ and the blue line is that portion of $y = -\sin(x)$, with $\imath \leq x \leq \frac{3\pi}{2}$. The green line is the corresponding part of $y = 1$. Thus, the volume in question is:

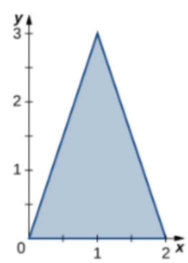
$$\int_{\frac{\pi}{2}}^{\pi} \int_{sin(x)}^{2} y^3\ dy\ dx + \int_{\pi}^{\frac{3\pi}{2}} \int_{-sin(x)}^{1} y^3\ dy\ dx.$$

To calculate these integrals, we will need the formula $\sin^4(x) = \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$, which can be derived from the double angle formulas for sine and cosine. For the first of the two integrals we have

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi}\int_{sin(x)}^{2} y^3\ dy\ dx &= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} y^4\Big|_{y=\sin(x)}^{y=1}\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} 1 - \sin^4(x)\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} 1 - (\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x))\ dx \\
&= \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} \frac{5}{8} + \frac{1}{2}\cos(2x) - \frac{1}{8}\cos(4x)\ dx \\
&= \frac{1}{4}(\frac{5}{8}x + \frac{1}{4}\sin(2x) - \frac{1}{32}\sin(4x))\Big|_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{4}\{(\frac{5}{8}\pi + 0 - 0) - (\frac{5}{8}\cdot\frac{\pi}{2} + 0 - 0)\} \\
&= \frac{5\pi}{64}.
\end{aligned}$$

Either by symmetry or essentially the same calculation, the second integral also equals $\frac{5\pi}{64}$. Thus the required volume is $\frac{5\pi}{64} + \frac{5\pi}{64} = \frac{5\pi}{32}$.

5. OS Chapter 5: #389: This problem asks to find the area of the triangle R:



,

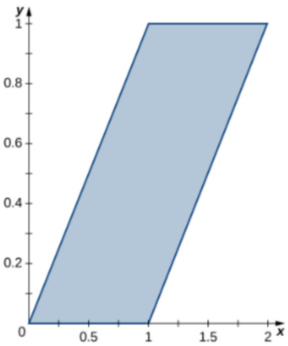
by finding a linear transformation T from the uv plane such that $T(0,0) = (0,0), T(1,0) = (2,0)$, and $T(0,1) = (1,3)$. This transformation will then take the triangle S in the uv-plane with vertices (0,0), (1,0), (0,1) to R.

Solution. From class we seen that we can take $T(u,v) = (2u+v, 3v)$. It is easy to check that $\text{Jac}(T) = -3$, so that $|\text{Jac}(T)| = 3$. Thus,

$$\begin{aligned}
\text{area}(R) &= \int\int_R\ dA \\
&= \int\int_S 3\ du\ dv \\
&= 3\cdot\text{area}(S) \\
&= 3,
\end{aligned}$$

as expected.

5. OS Chapter 5: #391. Calculate $\int\int_R (y^2 - xy)\, dA$, for R



,

for the given transformation.

Solution. The equations $u = y - x$ and $v = y$, can be rewritten as $x = v - u$ and $y = v$, which tells us our transformation should be $T(u, v) = (v - u, v)$. Substituting the vertices of R into the equations $u = y - x, v = y$ yields, vertices (0,0), (-1,0), (-1,1), (0,1) in the uv-plane, so that T transforms the rectangle $S = [-1, 0] \times [0, 1]$ in the uv-plane to R in the xy-plane. IT is easy to see that $\text{Jac}(T)| = 1$, so that

$$\begin{aligned}
\int\int_R (y^2 - xy)\, dA &= \int_0^1 \int_0^1 vu\, dv\, du \\
&= \int_0^1 \frac{u}{2}\, du \\
&= \frac{1}{4}.
\end{aligned}$$

5. OS Chapter 5: #431. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 16$, from $z = 1$ to $x + z = 2$.

Solution. We are finding the volume of the solid between the planes $z = 1$ and $z = 2 - x$, above the disk $D : 0 \leq x^2 + y^2 \leq 16$ in the xy-plane. Notice that if $x \geq 1$, then $2 - x \leq 1$ and if $x \leq 1$, then $1 \leq 2 - x$. Thus, the volume we seek is:

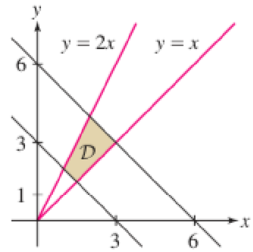
$$\int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2 - x) - 1\, dy\, dx + \int_{1}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1 - (2 - x)\, dy\, dx \qquad (\star)$$

For the first integral in $(\star)$ we have

$$\begin{aligned}
\int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (2 - x) - 1\, dy\, dx &= \int_{-4}^{1} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 1 - x\, dy\, dx \\
&= \int_{-4}^{1} (1 - x)y \Big|_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\, dx \\
&= 2\int_{-4}^{1} (1 - x)\sqrt{16 - 4x^2}\, dx \\
&\approx 71.78,
\end{aligned}$$

the last single integral being worked numerically, though one could use the standard (complicated) formula for $\int \sqrt{1 - x^2}\, dx$ typically found on the inside cover of a calculus book. Similarly, second integral in $(\star)$ is approximately 21.51, so the required area is approximately 93.29.

2. Calculate $\int\int_D (x+y)\ dA$, for D



,

using the transformation $G(u,v) = (\frac{u}{v+1}, \frac{uv}{v+1})$.

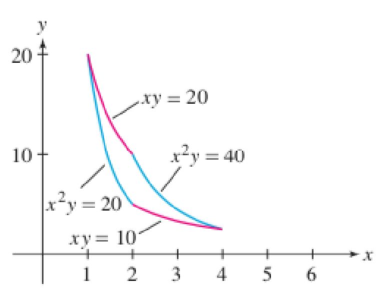
Solution. We need to find the region R in the uv-plan that $G(u,v)$ transforms to D. We use the equations of the lines bounding D. If $y = x$, then $\frac{u}{v+1} = \frac{uv}{v+1}$, from which we get $v = 1$. Similarly, the equation $y = 2x$ yields $v = 2$. The line in the xy plane containing (0,3) and (3,0) is $y = -x+3$. If we solve the corresponding equation $\frac{uv}{v+1} = -\frac{u}{v+1} + 1$ for u we get $u = 3$. Similarly, the lime through (0,6) and (6,0) in the xy plane gives rise to $u = 6$. Thus, the region R in the uv-plane is bounded by the lines $v = 1, v = 2, u = 3, u = 6$, so that $R = [3,6] \times [1,2]$. Calculating the Jacobian, we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u}{(v+1)^2} \end{pmatrix} = \frac{u}{(v+1)^3} + \frac{uv}{(v+1)^3} = \frac{u}{(v+1)^2}.$$

Since $3 \le u \le 6$, we have $|\frac{\partial(x,y)}{\partial(u,v)}| = \frac{u}{(v+1)^2}$. Thus,

$$\begin{aligned} \int\int_D (x+y)\ dA &= \int_3^6 \int_1^2 (\frac{u}{v+1} + \frac{uv}{v+1}) \cdot \frac{u}{(v+1)^2}\ dv\ du \\ &= \int_3^6 \int_0^1 \frac{u^2}{(v+1)^2}\ dv\ du \\ &= \int_3^6 u^2(-\frac{1}{v+1})_{v=1}^{v=2}\ du \\ &= \frac{1}{6}\int_3^6 u^2\ du \\ &= \frac{1}{6}(\frac{6^3}{3} - \frac{3^3}{3}) \\ &= \frac{21}{2}. \end{aligned}$$

3. Calculate $\int\int_D e^{xy}\ dA$, for D the region



,

by using the inverse of the transformation $F(x,y) = (xy, x^2y)$. Explain carefully how you obtain the domain of integration in the uv-plane

Solution. To find $G(u,v)$, the inverse of $F(x,y)$, we use the equations $u = xy$ and $v = x^2y$ to solve for x and y in terms of u and v. These equations give $\frac{u}{x} = y = \frac{v}{x^2}$, and thus, $\frac{u}{x} = \frac{v}{x^2}$ yields $x = \frac{v}{u}$. Since

$y = \frac{u}{x}$, we infer $y = \frac{u^2}{v}$. Thus, $G(u,v) = (\frac{v}{u}, \frac{u^2}{v})$. Note that when $xy = 10$ and $xy = 20$, then $u = 10$ and $u = 20$. This shows that $G(u,v)$ takes the lines $u = 10$ and $u = 20$ in the uv-plane to the hyperbolas $xy = 10$ and $xy = 20$ in the xy-plane. Similarly, $G(u,v)$ takes the lines $v = 20$ and $v = 40$ in the uv-plane to the graphs of $x^2y = 20$ and $x^2y = 40$ in the xy-plane. Now let's look at the four corners of the rectangle R in the uv-plane determined by the lines $u = 10, u = 20, v = 20, v = 40$. The lower left corner is (10, 20). $G(10,20) = (2,5)$ which is the lower left corner of the region D. $G(10,40) = (4,2.5)$ which is the lower right corner of D. Similarly, $G(u,v)$ takes the other two corners of R to the remaining corners of D, so it follows that G transforms R into D (by continuity of $G(u,v)$ and the fact that for the point (10, 30) in the interior of R, $G(10,30) = (3, \frac{10}{3})$ lies in the interior of D).

For the absolute value of the Jacobian of $G(u,v)$ we have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ \frac{2u}{v} & -\frac{u^2}{v^2}\end{pmatrix}\right| = \left| -\frac{1}{v}\right| = \frac{1}{v}.$$

Thus,

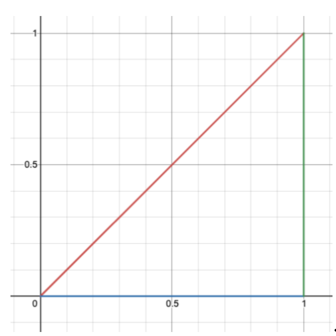
$$\begin{aligned}\int\int_D e^{xy}\ dA &= \int_{20}^{40}\int_{10}^{20} e^u \cdot \frac{1}{v}\ du\ dv \\ &= \int_{20}^{40} (e^{20} - e^{10}) \cdot \frac{1}{v}\ dv \\ &= (e^{20} - e^{10}) \int_{20}^{40} \frac{1}{v}\ dv \\ &= (e^{20} - e^{10}) \cdot (\ln(40) - \ln(20)) = (e^{20} - e^{10}) \cdot \ln(2).\end{aligned}$$

4. $\int\int_D \sqrt{x+y}(x-y)^2\ dA$, where D is the region bounded by the lines $x = 0, y = 0.x + y = 1$.

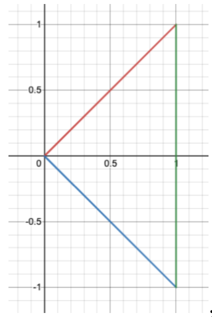
Solution. Because the integrand has no obvious ant-derivative with respect to either variable, we try to simplify it with a change of variables. If we choose u and v so that $u = x + y$ and $v = x - y$, then integrand then becomes $\sqrt{u}v^2$, which we can anti-differentiate. We can solve the system of equations $u = x + y$ and $v = x - y$ for x and y in terms of u and v and this will give the required change of variables. Upon doing so, we have $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Call this transformation $G(u,v)$. From this, it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix} = -\frac{1}{2},$$

from which we get $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$. We now have to see what region in the uv-plane gets transformed to the region D in the xy plane, which is the triangle below:



One edge of the triangle D is $x + y = 1$. In terms of u and v, this equation becomes $u = 1$. Thus, $G(u,v)$ transforms the line $u = 1$ in the uv plane to the line $x + y = 1$ in the xy-plane. Similarly, the equation $x = 0$ in terms of u and v becomes $u = y$, $v = -y$, so that $v = -u$, while the equation $y = 0$ yields $u = x, v = x$, so that $v = u$. Thus, if we let D_0 be the region in the uv-plane bounded by the lines $u = 1$, $v = -u$, and $v = u$,



,

we see that $G(D_0) = D$. Thus,

$$\begin{aligned}
\int\int_D \sqrt{x+y}(x-y)^2 \ dA &= \int\int_{D_0} \sqrt{u}v^2 \ \frac{1}{2} \ dA \\
&= \frac{1}{2}\int_0^1 \int_{-u}^{u} \sqrt{u}v^2 \ dv \ du \\
&= \frac{1}{2}\int_0^1 \sqrt{u}(\frac{v^3}{3})_{v=-u}^{v=u} \ du \\
&= \frac{1}{6}\int_0^1 2u^{\frac{7}{2}} \ du \\
&= \frac{1}{3}\cdot\frac{2}{9}(u^{\frac{9}{2}})\Big|_0^1 \\
&= \frac{2}{27}.
\end{aligned}$$

5. $\int\int_D \frac{1}{(x^2+y^2)^{\frac{3}{4}}} \ dA$, where D is the disk centered at the origin in $\mathbb{R}^2$ with radius R.

Solution. This is an improper double integral, as $f(x,y)$ is unbounded on D (since $\lim_{(x,y)\to(0,0)} f(x,y)$ tends to infinity). Let D_ϵ denote the region $\epsilon^2 \le x^2+y^2 \le R^2$, and we consider $\lim_{\epsilon\to 0}\int\int_{D_\epsilon} f(x,y) \ dA$. If this limit exists, it equals $\int\int_D \frac{1}{(x^2+y^2)^{\frac{3}{4}}} \ dA$. We have

$$\begin{aligned}
\lim_{\epsilon\to 0}\int\int_{D_\epsilon} f(x,y) \ dA &= \lim_{\epsilon\to 0}\int\int_{D_\epsilon} \frac{1}{(x^2+y^2)^{\frac{3}{4}}} \ dA \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R \frac{1}{(r^2)^{\frac{3}{4}}} \ r \ dr \ d\theta \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R \frac{1}{r^{\frac{3}{2}}} \ r \ dr \ d\theta \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^R r^{-\frac{1}{2}} \ dr \ d\theta \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi} 2\sqrt{r}\Big|_\epsilon^R \ d\theta \\
&= \lim_{\epsilon\to 0}\int_0^{2\pi} 2(\sqrt{R}-\sqrt{\epsilon}) \ d\theta \\
&= \lim_{\epsilon\to 0} 4\pi(\sqrt{R}-\sqrt{\epsilon}) \\
&= 4\pi\sqrt{R}.
\end{aligned}$$

6. $\int\int_{\mathbb{R}^2} e^{-x^2-y^2} \ dA$.

Solution. Letting S_R denote the sphere for radius R centered at the origin, we have

$$\begin{aligned}
\int\int_{\mathbb{R}^2} e^{-x^2-y^2}\ dA &= \lim_{R\to\infty} \int\int_{S_R} e^{-x^2-y^2}\ dA \\
&= \lim_{R\to\infty} \int_0^{2\pi}\int_0^R e^{-r^2} r\ dr\ d\theta \\
&= 2\pi \lim_{R\to\infty} \int_0^R e^{-r^2}\ r\ dr \\
&= 2\pi \lim_{R\to\infty} \frac{1}{2} e^{-r^2}\Big|_0^R \\
&= 2\pi \lim_{R\to\infty} \{-\frac{1}{2}e^{-R^2} + \frac{1}{2}\} \\
&= \pi.
\end{aligned}$$

7. $\int\int_D \frac{1}{x^2y^2}\ dA$, where D is the set of points in $\mathbb{R}^2$ satisfying $2 \le x \le \infty$ and $2 \le y \le \infty$.

Solution. We may test convergence of the double integral by integrating increasing rectangles (or squares) whose lower left corner is (2,2). Let D_a denote the square $[2,a] \times [2,a]$ with $2 \le a < \infty$. If the limit exists as $a \to \infty$, it equals $\int\int_D \frac{1}{x^2y^2}\ dA$.

$$\begin{aligned}
\lim_{a\to\infty} \int\int_{D_a} \frac{1}{x^2y^2}\ dA &= \lim_{a\to\infty} \int_2^a \int_2^a \frac{1}{x^2y^2}\ dy\ dx \\
&= \lim_{a\to\infty} \int_2^a -\frac{1}{x^2y}\Big|_{y=2}^{y=a}\ dx \\
&= \lim_{a\to\infty} \int_2^a -\frac{1}{ax^2} + \frac{1}{2x^2}\ dx \\
&= \lim_{a\to\infty} (\frac{1}{ax} - \frac{1}{2x})\Big|_{x=2}^{x=a} \\
&= \lim_{a\to\infty} \{(\frac{1}{a^2} - \frac{1}{2a}) - (\frac{1}{2a} - \frac{1}{4})\} \\
&= \frac{1}{4}
\end{aligned}$$

8. Compare your answer in problem 7 with $(\int_2^\infty \frac{1}{x^2}\ dx)^2$. Can you explain the relation between these two answers?

Solution. A calculation similar, though easier, than the one above shows that $\lim_{a\to\infty} \int_2^a \frac{1}{x^2}\ dx = \frac{1}{2}$. The answer in problem 12 is the square of the answer in problem 11, since

$$\begin{aligned}
\int_2^a \int_2^a \frac{1}{x^2y^2}\ dy\ dx &= \int_2^a \{\int_2^a \frac{1}{x^2y^2}\ dy\}\ dx \\
&= \int_2^a \frac{1}{x^2}\{\int_2^a \frac{1}{y^2}\ dy\}\ dx \\
&= \{\int_2^a \frac{1}{y^2}\ dy\} \int_2^a \frac{1}{x^2}\ dx \\
&= \{\int_2^a \frac{1}{y^2}\ dy\}^2,
\end{aligned}$$

and the limit of a square is the square of the limits, assuming both limits exist.

9. OS, Section 5.4: # 233, 241, 245, 281.

233. Solution: The key point is to insure that the plane $z + y + z = 9$ does not intersect domain in the

xy-plane. The required triple integral is

$$\int_0^2 \int_{x^2+1}^{7-x} \int_0^{9-x-y} dx\ dy\ dx.$$

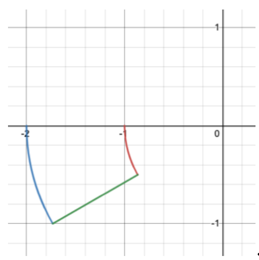
241. The required triple integral in cylindrical coordinates is

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^1 z \cdot r\ dz\ dr\ d\theta.$$

245. In cylindrical coordinates, the integral is

$$\int_\pi^\theta \int_1^2 \int_2^3 e^r \cdot r\ dz\ dr\ d\theta,$$

where θ is the upper bound of the polar region



Using that $x = \sqrt{3}y$ and $x^2 + y^2 = 1$ (say), the intersection of the line with the circle of radius one, occurs when $x = -\frac{\sqrt{3}}{2}$ and $= -\frac{1}{2}$, so that $\theta = \frac{7\pi}{6}$.

281. The equation of the sphere can be re-written as $x^2 + y^2 + (z-1)^2 = 1$, which in spherical coordinates becomes $\rho = 2\cos(\theta)$. As in previous examples finding the volume between a sphere and a cone, we need the angle the cone makes with the z-axis. The cone is easily seen to be a 45 degree cone, so that $0 \leq \phi \leq \frac{\pi}{4}$. Thus, the required triple integral is

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\cos(\phi)} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta.$$

In cylindrical coordinates, the cone is $z = r$ and the sphere is $z = \sqrt{1-r^2}+1$. Setting these equations equal to each other gives $r = 1$, which means the domain of integration in the xy-plane is the unit circle centered at the origin. Thus, in cylindrical coordinates, the required integral is

$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{1-r^2}+1} r\ dz\ dr\ d\theta.$$

10. Calculate $\int\int\int_B y^2 z^2\ dV$ for B the solid bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

Solution. If we let D denote the unit disk in the yz-plane, then

$$\begin{aligned}
\int\int\int_B y^2z^2\ dV &= \int\int_D\int_0^{1-y^2-z^2} y^2z^2\ dx\ dA \\
&= \int\int_D (1-y^2-z^2)y^2z^2\ dA \\
&= \int_0^{2\pi}\int_0^1 (1-r^2)(r\cos(\theta))^2(r\sin(\theta))^2\ rdrd\theta \\
&= \int_0^{2\pi}\int_0^1 (r^5-r^7)\cos^2(\theta)\sin^2(\theta)\ drd\theta \\
&= (\frac{1}{6}-\frac{1}{8})\int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)\ d\theta \\
&= \frac{1}{24}\int_0^{2\pi} \frac{1}{8}-\frac{1}{8}\cos(4\theta)\ d\theta \text{ (double angle formula twice)} \\
&= \frac{1}{24}\cdot\{\frac{\theta}{8}-\frac{1}{32}\sin(4\theta)\}_0^{2\pi} \\
&= \frac{1}{24}\cdot\frac{2\pi}{8} \\
&= \frac{\pi}{96}.
\end{aligned}$$

11. Calculate $\int\int\int_B z^3\sqrt{x^2+y^2+z^2}\ dV$, for B the solid hemisphere with radius 1 and $z \geq 0$.

Solution. Using spherical coordinates,

$$\begin{aligned}
\int\int\int_B z^3\sqrt{x^2+y^2+z^2}\ dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\cos(\phi))^3\sqrt{\rho^2}\rho^2\sin(\phi)\ d\rho\phi d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}}\int_0^1 \rho^6\cos^3(\phi)\sin(\phi)\ d\rho d\phi \\
&= \frac{2\pi}{7}\int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\ d\phi \\
&= \frac{2\pi}{7}\cdot(-\frac{\cos^4(\theta)}{4})\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{14}.
\end{aligned}$$

12. These problems were selected at random because they looked interesting. It turns out that computationally they are difficult, for no apparent reason other than making the computation difficult. Answers are given (mostly) in the handwritten pages that follow. Lagarange multiplier problems on the exam will not be so computationally difficult.